Exam 1 (Take-Home Portion)

Your Name:

Names of Any Collaborators:

Instructions

This portion of Exam 1 is worth a total of 22 points and is worth 30% of your overall score on Exam 1. This take-home exam is due by 5PM **Tuesday, February 19**. Your overall score on Exam 1 is worth 20% of your overall grade. Good luck and have fun!

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems or problems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem X or Problem Y, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

The following problems are all about the Catalan numbers, which are named after Eugène Catalan. Although these are now known as Catalan numbers, Catalan called them Segner numbers, after Johann Segner. Segner, in turn, learned of the sequence from Euler! But even Euler's work was predated by that of Mongolian mathematician Minggatu, though it seems unlikely that Euler was aware of Minggatu. The sequence of Catalan numbers is among the most famous sequences in mathematics. The Catalan numbers have a remarkable ability to pop up in surprising locations. Some combinatorialists joke that a combinatorics paper is not complete until the Catalan numbers have made an appearance.

A permutation $w = w(1)\cdots w(n)$ in S_n is said to contain the pattern 231 if there is a triple of indices $i < j < k$ such that $w(k) < w(i) < w(j)$. Otherwise we say w is *231-avoiding*. Intuitively, a permutation is 231-avoiding if it does not contain a subsequence (read from left to right) of the form "medium", "high", "low". Let $S_n(231)$ denote the set of 231-avoiding permutations in S_n.

For example, the permutation 53412 contains the pattern 231 in two different ways, with $(i,j,k) = (2,3,4)$ and with $(i,j,k) = (2,3,5)$. So, 53412 is not 231-avoiding.

The *nth Catalan number*, denoted C_n, counts the number of 231-avoiding permutations in S_n, That is,

$$C_n = |S_n(231)|.$$

By convention $C_0 = 1$, since there is one empty permutation that trivially avoids the pattern.

A *Dyck path* (named after German mathematician Walther Franz Anton von Dyck) of length $2n$ is a lattice path from $(0,0)$ to (n,n) that takes n steps "East" from (i,j) to $(i+1,j)$ and n steps "North" from (i,j) to $(i,j+1)$, such that all points on the path satisfy $i \leq j$. In other words, when drawn in the cartesian plane, a Dyck path lies on or above the line $y = x$. The set of all Dyck paths of length $2n$ is denoted $\mathrm{Dyck}(n)$. For this exam, let $d_n := |\mathrm{Dyck}(n)|$.

1. (2 points) Compute C_1, C_2, and C_3 via brute-force by listing all of the appropriate permutations.

2. (2 points) Compute d_1, d_2, and d_3 via brute-force by drawing all of the appropriate Dyck paths.

3. (4 points) By considering different values of i such that $w(i+1) = n$, show that $S_n(231)$ can be partitioned into subsets that are each in bijection with $S_i(231) \times S_{n-i-1}(231)$. Use your correspondence to show that for any $n \geq 1$, we have

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1},$$

where $C_0 = 1$.* *Warning:* In your argument, don't forget to explain why certain permutations are 231-avoiding.

4. (4 points) Prove that $d_n = C_n$ by describing a bijection between $\mathrm{Dyck}(n)$ and $S_n(231)$. There are a few different ways to attack this problem, but one way is to think of 231-avoiding permutations as non-attacking rook arrangements. See if you can find a natural correspondence to a Dyck path for a given non-attacking rook arrangement for a 231-avoiding permutation.

*This shows that the Catalan numbers form the sequence from part (4) of Problem 54.

5. Let's find a closed form for d_n.
   (a) (2 points) Argue that the number of lattice path (not just Dyck paths) from $(0,0)$ to (n,n) is equal to $\binom{2n}{n}$.
   (b) (2 points) Argue that the number of lattice paths from $(0,0)$ to $(n+1,n-1)$ is equal to $\binom{2n}{n-1}$.
   (c) (2 points) Prove that there is a bijection from the set of lattice paths from $(0,0)$ to (n,n) that pass below $y=x$ is in bijection with the set of lattice paths from $(0,0)$ to $(n+1,n-1)$. *Hint:* Consider the first point on lattice path from $(0,0)$ to (n,n) that passes below $y=x$. Reflect the remaining portion of the path over the appropriate line to get a path from $(0,0)$ to $(n+1,n-1)$.
   (d) (2 points) Prove that $d_n = \binom{2n}{n} - \binom{2n}{n-1}$.
6. (2 points) Prove that $\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$.

Boom! We just proved some really cool stuff. Here's a summary:

- There is a bijection between 231-avoiding permutations in S_n and Dyck paths from $(0,0)$ to (n,n) and both are counted by the Catalan numbers.
- The Catalan numbers satisfy the following recurrence:

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1},$$

  where $C_0 = 1$.
- The Catalan numbers have a nice closed form:

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

It turns out that there are over two hundred different sets of combinatorial objects that are enumerated by Catalan numbers. Here's a partial list:

- 123-avoiding permutations
- 321-avoiding permuations
- Noncrossing partitions
- Balanced parenthesizations
- Two-row standard Young tableaux
- Decreasing binary trees
- Triangulations of a polygon

We'll tackle each of these and likely more later in the semester.