**Your Name:**

**Names of Any Collaborators:**

## Instructions

This portion of the Final Exam is worth a total of 16 points and is due by **5pm on Thursday, May 12**. Your total combined score on the in-class portion and take-home portion is worth 20% of your overall grade.

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem xyz, then you should say so.
2. Unless you prove them, you cannot use any results that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

**I will vigorously pursue anyone suspected of breaking these rules.**

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

**Signature:**

Good luck and have fun!

Complete any **four** of following problems. Each problem is worth 4 points. Write your solutions on your own paper and please put the problems in order. Assume $R$ is a ring with 1 and $M$ is a left $R$-module.

1. Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of submodules of $M$. Prove that $\cup_{i=1}^{\infty} N_i$ is a submodule of $M$.

2. An element $m$ of the $R$-module $M$ is called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted
$$\mathrm{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$
Prove that if $R$ is an integral domain, then $\mathrm{Tor}(M)$ is a submodule of $M$ (called the *torsion* submodule of $M$.

3. Let $I$ be an ideal of $R$ and let $M'$ be the subset of elements $a \in M$ that are annihilated by some power, $I^k$, of the ideal $I$ (where the power may depend on $a$). Prove that $M'$ is a submodule of $M$.

4. Let $V = \mathbb{R}^2$ and let $T$ be the linear transformation from $V$ to $V$ which is rotation clockwise about the origin by $\pi/2$ radians. Prove the $V$ and 0 are the only $\mathbb{R}[x]$-submodules for $T$.

5. Let $R$ be a commutative ring. Prove that $\mathrm{Hom}_R(R, M)$ and $M$ are isomorphic as left $R$-modules.

6. Let $\phi : M \to N$ be an $R$-module homomorphism. Prove that $\phi(\mathrm{Tor}(M)) \subseteq \mathrm{Tor}(N)$.

7. Prove that if $A$ and $B$ have the same cardinality, then the free $R$-modules $F(A)$ and $F(B)$ are isomorphic as $R$-modules.

8. Prove that if $M$ is a finitely generated $R$-module that is generated by $n$ elements, then every quotient of $M$ may be generated by $n$ (or fewer) elements, and in particular, if $M$ is cyclic, then every quotient of $M$ is cyclic.

9. An $R$-module $M$ is called *irreducible* if $M \neq \emptyset$ and 0 and $M$ are the only submodules of $M$. Prove that $M$ is irreducible iff $M \neq \emptyset$ and $M$ is a cyclic module with any nonzero element as a generator.