Homework 1

Abstract Algebra II

Complete the following problems. Note that you should only use results that we've discussed so far this semester or last semester.

Problem 1. Determine whether each of the following statements is 'True' or 'False'. If a statement is False, provide a counterexample. If a statement is True, then either provide a reference or a proof.

(a) The image of a ring with 10 elements under some ring homomorphism may consist of 3 elements.

(b) Every subring of an integral domain is an integral domain.

(c) If R is an integral domain but not a field, then it is possible that R contains a subring that is a field.

(d) Every ring consisting of exactly 5 elements is a field.

(e) If R is a finite commutative ring with 1 having no zero divisors, then R is a field.

(f) Suppose R is an integral domain R. Then R/I is a field iff I is a prime ideal.

(g) The set of matrices

$$F = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

with entries from the field $\mathbb{Z}_2$ is a field, under ordinary matrix addition and multiplication.

(h) Suppose R is a commutative ring. If $ra \in I$ for all $r \in R$ and $a \in I$, then I is a two-sided ideal in R.

Problem 2. Let R be a commutative ring with 1 and let $U(R)$ be the group of units in R. Prove that R has a unique maximal ideal iff $R \setminus U(R)$ is an ideal. *Note:* You may assume that maximal ideals exist.

Problem 3. A **simple ring** is a ring with no nonzero proper 2-sided ideals. If R is a ring, then the **center** of R is defined to be $Z(R) := \{x \in R \mid rx = xr \text{ for all } r \in R\}$. Prove that the center of a simple ring with 1 is a field. *Note:* You must first show that the center is a subring.

Problem 4. Assume R is a commutative ring with 1. Prove that the ideal (x) in $R[x]$ is a maximal ideal iff R is a field.

Problem 5. Assume R is a commutative ring with $1 \neq 0$ and for each $r \in R$, there exists an integer $n > 1$ such that $r^n = r$. Prove that every prime ideal of R is maximal.

Problem 6. Assume F is a finite field. Prove that there exists a prime p such that all non-zero elements of F have an additive order of p.

Problem 7. Assume R is a Euclidean Domain. Let m be the minimum integer in the set of norms of nonzero elements of R. Prove that every nonzero element of R of norm m is a unit. Deduce that a nonzero element of norm zero is a unit.

Problem 8. Assume R is a Euclidean Domain. Prove that if $(a, b) = 1$ and a divides bc, then a divides c. More generally, prove that if a divides bc with nonzero a, then $a/(a, b)$ divides c. *Note:* I don't care what order you do these in. Certainly, the general statement handles the case when $(a, b) = 1$. However, you may find it useful to do the special case first.