Homework 2

Abstract Algebra II

Complete the following problems. Note that you should only use results that we've discussed so far this semester or last semester.

Problem 1. Let R be a Euclidean Domain with norm N satisfying $N(a) \leq N(ab)$ for all nonzero $a, b \in R$[1]. Prove that $a \in R$ is a unit iff $N(a) = N(1)$. *Note:* Actually, I don't think we need this extra requirement on N, but since I already modified the problem, I'll just leave it.

Problem 2. Consider the Euclidean Domain $\mathbb{Z}[i]$ with norm given by $N(a+bi) = a^2 + b^2$.

(a) Find the units in $\mathbb{Z}[i]$.

(b) For each of the following pairs, find q and r such that $a = bq + r$ with $r = 0$ or $N(r) < N(b)$.

  (i) $a = 11 + 8i, b = 1 + 2i$

  (ii) $a = -17 + 15i, b = 3 + i$

Problem 3. Consider the Euclidean Domain $\mathbb{Q}[x]$ with norm given by $N(p(x)) = \deg(p(x))$.

(a) Prove that $(x^2 + 1, x^3 + 1) = \mathbb{Q}[x]$.

(b) Find polynomials $a(x)$ and $b(x)$ such that $(x^2 + 1)a(x) + (x^3 + 1)b(x) = 1$.

Problem 4. Let R be an integral domain and let u be a unit of R.

(a) Prove that if $p \in R$ is prime, then up is prime.

(b) Prove that if $p \in R$ is irreducible, then up is irreducible.

Problem 5. Consider the ring $\mathbb{Z}[\sqrt{-5}]$ with norm $N(a + b\sqrt{-5}) = a^2 + 5b^2$.

(a) Justify my claim in Example 1.85(3) that $6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ are two distinct factorizations of 6 into irreducibles in $\mathbb{Z}[\sqrt{-5}]$. *Hint:* Start by showing that $N(xy) = N(x)N(y)$ for all $x, y \in \mathbb{Z}[\sqrt{-5}]$.

(b) Prove that $1 + \sqrt{-5}$ is not prime in $\mathbb{Z}[\sqrt{-5}]$.

Problem 6. Consider the ring $\mathbb{Z}[2\sqrt{2}]$.

(a) Prove that $\mathbb{Z}[2\sqrt{2}]$ is not a UFD. *Hint:* Fiddle around with 8. You will need to justify that certain ring elements are irreducibles. One way to do this is to play with the map $N : \mathbb{Z}[2\sqrt{2}] \to \mathbb{Z} \cup \{0\}$ given by $N(a + 2b\sqrt{2}) = |a^2 - 8b^2|$ (the vertical bars denote absolute value). It would be useful to know $N(rs) = N(r)N(s)$, $N(r) = 1$ iff r is a unit, and $N(r) \neq 2$ for all $r \in R$. If you want to use these facts, you should prove them.

(b) If possible, give an example of an ideal of $\mathbb{Z}[2\sqrt{2}]$ that is not principal. If not possible, briefly explain why.

[1] This extra requirement on N is sometimes part of the definition of Euclidean Domain.