Homework 8

Abstract Algebra II

Complete the following problems. Note that you should only use results that we've discussed so far this semester or last semester.

Problem 1. Let K/F be a field extension, where F is a perfect field. If $f(x) \in F[x]$ has no repeated irreducible factors, prove that $f(x)$ has no repeated irreducible factors over K.

Problem 2. Prove that there are only a finite number of roots of unity in any finite extension K of $\mathbb{Q}$. *Note:* You may use the fact that $[\mathbb{Q}(\zeta):\mathbb{Q}] = \phi(k)$, where $\zeta \in K$ is a primitive kth root of unity.

Problem 3. Let $\tau : \mathbb{C} \to \mathbb{C}$ be defined via $\tau(a+bi) = a - bi$ (complex conjugation).

(a) Prove that τ is an automorphism of $\mathbb{C}$.

(b) Determine the fixed field of τ.

Problem 4. Prove that $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are not isomorphic.

Problem 5. Determine the automorphisms of the extension $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}(\sqrt{2})$ explicitly.

Problem 6. Consider $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$.

(a) Prove that any $\sigma \in \mathrm{Aut}(\mathbb{R}/\mathbb{Q})$ takes squares to squares and takes positive reals to positive reals. Conclude that $a < b$ imples $\sigma(a) < \sigma(b)$ for every $a, b \in \mathbb{R}$.

(b) Prove that

$$-\frac{1}{m} < a - b < \frac{1}{m} \text{ implies } -\frac{1}{m} < \sigma(a) - \sigma(b) < \frac{1}{m}$$

for every positive integer m. Conclude that σ is a continuous map on $\mathbb{R}$.

(c) Prove that any continuous map on $\mathbb{R}$ that is the identity on $\mathbb{Q}$ is the identity map, hence $\mathrm{Aut}(\mathbb{R}/\mathbb{Q})$

Problem 7. Consider the polynomial $p(x) = x^4 - 5x^2 + 6 \in \mathbb{Q}[x]$. Determine $\mathrm{Aut}(K/\mathbb{Q})$, where K is the splitting field of $p(x)$.