Homework 8

Combinatorics of Genome Rearrangements

You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in his or her own work. In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments with no questions asked**.

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing assignments for this course, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, the textbook, me, and your own intuition. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.** Please read NAU's Academic Integrity Policy.

Complete the following problems.

1. We can extend an ordinary permutation $\pi = \pi_1 \cdots \pi_n \in S_n$ to a circular permutation π° by inserting an extra element 0 as both predecessor of π_1 and successor of π_n, and taking the equivalence class under cyclic shifts. We will write

$$\pi^\circ = 0\pi_1 \cdots \pi_n,$$

but it is important to emphasize that π° represents an equivalence class. For example, if $\pi = 312$, under the action of cyclic shifts $0312 = 3120 = 1203 = 2031$, so officially $\pi^\circ = \{0312, 3120, 1203, 2031\}$. From a circular permutation π°, we uniquely retrieve the ordinary permutation π by removing the element 0 and letting its successor be the first element of π. A block transposition move on π has an effect on π° that is easy to state: The circle is cut into three segments that are then glued together in the other possible order. Now, we take things one step further. An m-**step cyclic value shift** of π° is defined via

$$m + \pi^\circ := m\ m + \pi_1\ \cdots m + \pi_n \pmod{n+1}.$$

The equivalence class of π° under such value shifts is the **toric permutation** $\pi^\circ_\circ$. For example, if $\pi = 312$, then the representatives of the toric permutation are:

$$\begin{aligned} \pi^\circ &= 0312 \\ 1 + \pi^\circ &= 1023 \\ 2 + \pi^\circ &= 2130 \\ 3 + \pi^\circ &= 3201 \end{aligned}$$

It follows that $312^\circ_\circ = 231^\circ_\circ = 132^\circ_\circ$.

   (a) Explain why $m + \mathrm{id}^\circ = \mathrm{id}^\circ$ for all m.

   (b) Prove that if an optimal block transposition sequence takes π° to id°, then the same sequence of moves takes $m + \pi^\circ$ to $m + \mathrm{id}^\circ$.

   (c) Explain why any optimal sequence of block transpositions for $\pi \in S_n$ can be translated into an optimal sequence of block transpositions for any representative of $\pi^\circ_\circ$.

2. Assume $n = 2k+1$ for $k \geq 1$ (so that n is odd and greater than or equal to 3) and consider the permutation $\pi = n \cdots 321$ (i.e., the permutation that is the reverse of the identity). Consider the following sequence of block transpositions on $\pi^\circ = n \cdots 3210$ (0 was intentionally written on the right) consisting of $k+1$ moves of the same type: a block of size two is moved k steps to the left. First $\boxed{k+1\ k}$ is moved to the far left, then $\boxed{k+2\ k-1}$ is inserted in the middle of the block last moved, etc. The last pair to be moved is $\boxed{n\ 0}$. For example, if $k = 3$ (so that $n = 7$), the algorithm yields:

$$\boxed{765}\boxed{43}210 \to 4\boxed{376}\boxed{52}10 \to 45\boxed{237}\boxed{61}0 \to 456\boxed{123}\boxed{70} \to 45670123.$$

In general, the result of the algorithm applied to π is $k+1 \cdots n\ 0 \cdots k$.

(a) Explain why $k+1 \cdots n\ 0 \cdots k$ is a representative of the toric identity permutation. That is, why is $k+1 \cdots n 0 \cdots k \in \mathrm{id}_\circ^\circ$?

(b) Explain why $d_{bt}(\pi) \leq k+1$.

(c) Explain why any single block transposition applied to π reduces the number of descents by 1.

(d) Explain why any single block transposition applied to the identity produces one descent.

(e) Recall a previous result that states that every block transposition decreases the number of descents by at most 2. Now, argue that $d_{bt}(\pi) \geq \frac{n-3}{2} + 2$.

(f) Explain why $d_{bt}(\pi) = k+1$.