Exam 1 (Part 2)

Your Name:

Names of Any Collaborators:

Instructions

Answer each of the following questions by 5:00PM on **Friday, March 8**. This part of Exam 1 is worth a total of 22 points and is worth 50% of your overall score on Exam 1. Your overall score on Exam 1 is worth 25% of your overall grade.

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts. Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing the following problems, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes ChatGPT, Chegg, and Course Hero. On the other hand, you may use each other, the textbook, me, and your own intuition. Further information:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.** To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

 Written by D.C. Ernst

Note: You are welcome to use any earlier result on this exam to prove a later result, even if you did not prove the earlier result.

Define gamegraphs G and H to be **equivalent**, written $\mathsf{G} \equiv \mathsf{H}$, if $o^+(\mathsf{G}+\mathsf{X}) = o^+(\mathsf{H}+\mathsf{X})$ for every gamegraph X. Loosely speaking, G and H are equivalent if G acts like H in any sum of games.

1. (4 points each) Complete **two** of the following.

   (a) Prove that equivalence of gamegraphs is an equivalence relation on the collection of gamegraphs (and hence does not have a dumb name).

   (b) Prove that $\mathsf{G} \equiv \text{NIM}(0)$ if and only if $o^+(\mathsf{G}) = P$.

   (c) Prove that if G and H are gamegraphs such that $o^+(\mathsf{H}) = P$, then $\mathsf{G} \equiv \mathsf{G} + \mathsf{H}$.

   (d) Prove that if G and H are gamegraphs, then $\mathsf{G} \equiv \mathsf{H}$ if and only if $o^+(\mathsf{G} + \mathsf{H}) = P$.

We need a new concept for the next problem. The **minimum excluded value** (or **mex** for short) of a set S of nonnegative integers, denoted $\text{mex}(S)$, is the least nonnegative integer not appearing in S. For example, $\text{mex}(\{0, 1, 2, 4, 5, 7\}) = 3$ and $\text{mex}(\emptyset) = 0$. For the rest of the exam, assume that every gamegraph G has the property $\text{Opt}(p)$ is finite for every position p of G. Everything that follows extends to ordinals in the expected way, but we'll just focus on this special case.

2. (8 points) Prove that if G is a gamegraph such that $\text{Opt}(p)$ is finite for every position p, then there exists $n \in \mathbb{N} \cup \{0\}$ such that $\mathsf{G} \equiv \text{NIM}(n)$.

   Hint: Use structural induction. Here's some additional guidance. Suppose s is the source of G and let $\text{Opt}(s) = \{p_1, \ldots, p_m\}$. Each position p_i corresponds to a gamegraph G_{p_i}. By induction, each $\mathsf{G}_{p_i} \equiv \text{NIM}(k_i)$ for some $k_i \in \mathbb{N} \cup \{0\}$. Define G' to be the gamegraph whose options from the starting position correspond to $\text{NIM}(k_1), \ldots, \text{NIM}(k_m)$. Our goal is to show that $\mathsf{G} \equiv \text{NIM}(n)$, where $n = \text{mex}(\{k_1, \ldots, k_m\})$. First, argue that $\mathsf{G} \equiv \mathsf{G}'$ by using Problem 1(d). Next, argue that $\mathsf{G}' \equiv \text{NIM}(n)$ (where $n = \text{mex}(\{k_1, \ldots, k_m\})$). Lastly, put a bow on everything by appealing to transitivity of $\equiv$.

The **nim-number** $\text{nim}(p)$ of a position p of a rulegraph R is defined recursively as the minimum excludant of the nim-numbers of the options of p. That is,

$$\text{nim}(p) := \text{mex}(\text{nim}(\text{Opt}(P))).$$

Warning: The subtle difference in notation between NIM and nim is both awesome and potentially confusing: $\text{NIM}(n)$ denotes the gamegraph for Nim played on n stones while $\text{nim}(p)$ is a nonnegative integer associated to a position p. The nim-number of a gamegaph G is the nim-number of its starting position. That is, $\text{nim}(\mathsf{G}) := \text{nim}(s)$, where s is the starting position of G. The following facts should be clear:

- If t is a terminal position, then $\text{nim}(t) = 0$.
- $\text{nim}(\text{NIM}(n)) = n$ for all $n \in \mathbb{N} \cup \{0\}$ (follows from induction).
- The nim-number function is a valuation with $\mu = \text{mex}$.

If needed, you are welcome to use these facts.

3. (4 points) Prove **one** of the following.

   (a) Prove that $\text{nim}(\mathsf{G}) = n$ if and only if $\mathsf{G} \equiv \text{NIM}(n)$.

   (b) Prove that if G and H are emulationally equivalent, then $\mathsf{G} \equiv \mathsf{H}$.

4. (2 points) Provide a counterexample to show that converse of Problem 3(b) is false.

