Homework 2

Combinatorial Game Theory

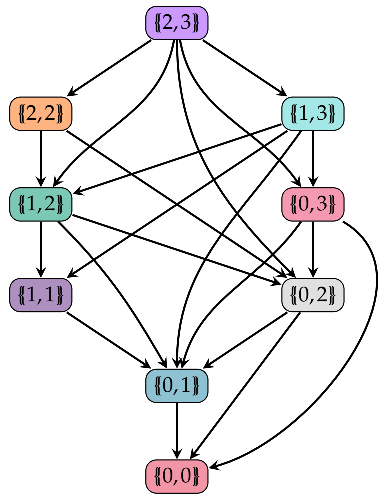
Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions on Discord. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

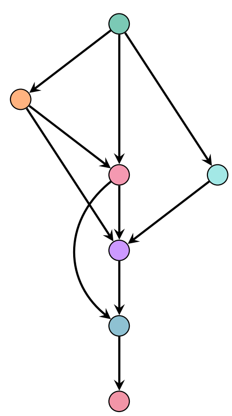
1. Provide examples of each of the following that differ from examples given in class.
   (a) A gamegraph G where normal play and misere have the same outcome.
   (b) A gamegraph G where normal play and misere have different outcomes.
2. Complete each of the following.
   (a) Draw the gamegraph for $\textsc{Nim}(2)$. Label the formal birthday of each position. Assuming normal play, also label each position as a P-position or N-position.
   (b) Draw the gamegraph for $\textsc{Nim}(3)$. Label the formal birthday of each position. Assuming normal play, also label each position as a P-position or N-position.
   (c) Draw the gamegraph for $\textsc{Nim}(2)+\textsc{Nim}(3)$. Label the formal birthday of each position. Assuming normal play, also label each position as a P-position or N-position.
3. Let R and S be rulegraphs and let G and H be gamegraphs.
   (a) Prove that $\mathsf{R}+\mathsf{S}$ is a rulegraph.
   (b) If (p,q) is a position in $\mathsf{R}+\mathsf{S}$, discover a formula for $\operatorname{fbd}(p,q)$ with justification.
   (c) Prove that $\mathsf{G}+\mathsf{H}$ is a gamegraph.
   (d) Provide an interpretation of what it means to play the game $\mathsf{G}+\mathsf{H}$.
   (e) OMITTED

4. We say that a digraph map $\alpha : C \to D$ is **option preserving** if $\mathrm{Opt}(\alpha(p)) = \alpha(\mathrm{Opt}(p))$ for each $p \in V(C)$. A gamegraph map $\alpha : \mathsf{G} \to \mathsf{H}$ is **source preserving** if it takes the starting position of G to the starting position of H. Source and option preserving gamegraph maps are playing the role of "game homomorphisms". Let H be the gamegraph given below. Define $\alpha : \textsc{Nim}(2) + \textsc{Nim}(3) \to \mathsf{H}$ via $\alpha(u,v) = \{\!\{u,v\}\!\}$. Verify that α is source preserving and option preserving.



H

5. Let K be the gamegraph given below. Find a surjective source and option preserving map from $\textsc{Grundy}(\{\!\{7\}\!\})$ to K. Briefly explain why your proposed map is surjective, source preserving, and option preserving.



K

6. Prove that if $\alpha : \mathsf{S} \to \mathsf{T}$ and $\beta : \mathsf{R} \to \mathsf{S}$ are source preserving rulegraph maps, then the composition $\alpha \circ \beta$ is also source preserving.

7. Prove that if $\alpha : \mathsf{S} \to \mathsf{T}$ and $\beta : \mathsf{R} \to \mathsf{S}$ are option preserving rulegraph maps, then the composition $\alpha \circ \beta$ is also option preserving.